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REGRESSION COMPONENT ANALYSIS

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Regression component decompositions (RCD) are defined as a special class of component decompositions where the pattern contains the regression weights for predicting the observed variables from the latent variables. Compared to factor analysis, RCD has a broader range of applicability, greater ease and simplicity of computation, and a more logical and straightforward theory. The usual distinction between factor analysis as a falsifiable model, and component analysis as a tautology, is shown to be misleading, since a special case of regression component decomposition can be defined which is not only falsifiable, but empirically indistinguishable from the factor model.

INTRODUCTION

In 1904, Charles Spearman proposed factor analysis as a falsifiable mathematical model for the description of intelligence tests. He adduced a fair amount of empirical evidence in support of his 'Two-factor Theory': 'The average inter-columnar correlation from the tables of fourteen different investigators, summarizing thirty years of psychological researches and representing a great wealth of test material, was unity, as expected by the unifocal hypothesis of a general factor. It seemed to be the most striking quantitative fact in the history of psychology' (Dodd, 1928, p. 214).

After a period of optimism and refinement of the methodology, a number of theoretical problems emerged which seemed to shed doubts on the stringency of Spearman's factor model. Thomson (1919) argued that other mathematical theories could be used to explain Spearman's data. In 1928, E. B. Wilson showed that the latent variables of the model, the factors, are not uniquely defined by the factor model ('factor indeterminacy'). In 1939 (Wilson & Worcester, 1939), he also showed that sometimes the variance parameters of the model are not identifiable ('identifiability problem').

During the 1940s, Thurstone, following a suggestion by Garnett (1919), successfully popularized a multiple factor extension of Spearman's theory. Multiple factor analysis became widely accepted as one of the most promising methodological advances of psychology. During this period most of the theoretical problems of the factor model were ignored. Factor analysis gradually lost its character as a model, and became more and more a tool for data reduction. Some of the theoretical problems of the factor model were recently brought back

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into focus in a number of papers (Schönemann, 1971; Schönemann & Wang, 1972; Steiger & Schönemann, 1975) which finally succeeded in reopening these outstanding questions. A very comprehensive and authoritative paper by Guttman in 1955 had been virtually ignored, together with all previous work in this area. In the empirical part of their 1972 paper, Schönemann & Wang reanalyzed the data of 13 published factor analyses, employing the recently perfected maximum likelihood algorithms for factor extraction, which make it feasible to test the model statistically. They found (1) that the factor model usually did not fit statistically for the small number of common factors which appeared in the published accounts and (2) that some of the common factors were usually poorly defined. Frequently, the correlation of a factor with a minimally correlated equivalent factor was zero. As Guttman (1955) had pointed out, this 'raises the question what is being estimated in the first place; instead of only one primary trait there are many widely different variables associated with a given profile of factor loadings'. Finally, they found (3) that both problems, factor indeterminacy and lack of identifiability, grew worse as the number of factors was raised in an effort to improve the statistical fit.

Understandably, this study has provoked some controversy about the merits of the factor model. One might expect some resistance against discarding factor analysis, because, in spite of the theoretical difficulties of the underlying model, the method has proven flexible and useful as a data reduction technique. In this paper, we propose and discuss an alternative method for data reduction which has many of the practical virtues of factor analysis, is equally flexible, but computationally more efficient and free from its theoretical problems.

1. Definitions

The basic problem is the description of p given random variables y_i in $\eta' = (y_1, \ldots, y_p)$ in terms of $m \leqslant p$ random variables x_j in $\xi' = (x_1, \ldots, x_m)$, and p residuals e_i in $\varepsilon' = (e_1, \ldots, e_p)$. In the factor analysis model,

$$\eta = A^* \xi^* + \varepsilon^*, \tag{1.1}$$

the m 'common factors' in ξ and the p 'unique factors' in ε are defined implicitly in terms of their variance and covariance behavior, viz.

$$\operatorname{var}\begin{pmatrix} \xi^* \\ \varepsilon^* \end{pmatrix} = \begin{bmatrix} \psi^* & \phi \\ \phi & U^2 \end{bmatrix} \tag{1.2}$$

with

$$\operatorname{rank}\left[\operatorname{var}\begin{pmatrix} \xi^* \\ \varepsilon^* \end{pmatrix}\right] = p + m. \tag{1.2a}$$

As is well known (e.g. Wilson, 1928; Guttman, 1955; Schönemann, 1971; Schönemann & Wang, 1972), this implicit definition of the factors of the factor model precludes the possibility of expressing them as linear combinations of the given random variables y_i. As a consequence, one is left with an indeterminacy at the random variable level which is still the subject of some controversy. This indeterminacy has led to the questionable practice of 'estimating' factors by various methods. The notion of 'estimating' a parameter that is not uniquely defined is somewhat unconventional. Advocates of such 'estimation' have yet to provide a clear rationale for their efforts.

To avoid these and other difficulties, we discard the problematic definition (1.2), and replace it by a definition of 'components' as linear combinations of the observed random variables, but retaining the decomposition (1.1). We thus propose:

Definition 1. m random variables x_j in $\xi' = (x_1, \ldots, x_m)$ will be called 'components' of the $p \geqslant m$ given random variables y_i in $\eta = (y_1, \ldots, y_p)$ if they can be written as linear combinations of the given y_i, i.e. iff there exists a matrix of 'defining weights' $B = (b_{ij})$ such that

$$\xi = B'\eta. \tag{1.3}$$

For convenience we assume that B is so chosen that the components x_j are linearly independent, i.e.

$$\operatorname{var}(\xi) = B'\Sigma B = \psi \tag{1.3a}$$

is positive definite. Once we have decided on a B in (1.3), ξ will be uniquely defined.

This is a rather broad definition. It covers 'principal components', 'canonical variates', 'linear discriminant functions', '(partial) images', 'anti-images', as well as any rotations thereof as special cases. It also covers the various kinds of 'factor score estimates'. These 'estimates' are of practical use primarily because they are expressed as linear combinations of the original variables, and thus uniquely defined in contrast to the indeterminate factors.

In statistical applications, the representation of the original variables in terms of the 'components' is often not the major goal. For example, linear discriminant functions are linear functions of the observed variables. They are defined for optimally predicting group membership and fit the definition of 'components'. If there are three groups, there will be at most two linear discriminant functions. Generally there is no interest in whether these components could be used to 'represent' adequately the given random variables. Such representation is not their purpose.

In psychometrics, and especially in factor analysis, on the other hand, such representation of random variables is often the major goal. This distinction in perspective should be maintained. Here, our main interest is in 'component decompositions', i.e. the description of a given random vector η in terms of components. We therefore propose:

Definition 2. A representation of η in terms of m components ξ will be called a 'component decomposition' (of η) iff η is written

$$\begin{aligned} \eta &= A\xi + \varepsilon = \hat{\eta} + \varepsilon \\ &= AB'\eta + (I - AB')\eta \end{aligned} \tag{1.4}$$

for some $p \times m$ 'pattern' of constants $A = (a_{ij})$ of full column rank m. The p random variables e_i in $\varepsilon' = (e_1, \ldots, e_p)$ will be called 'residuals' and the p random variables in $\hat{\eta} = A\xi$ 'predicted parts'.

This way of representing η in terms of ξ is analogous to the factor model (1.1) except that ξ, ε are linear functions of the observed variables, whereas the factors ξ^*, ε^* are not. An example of such a component decomposition is Guttman's Image Analysis. The predicted parts are the (partial) images and the residuals the anti-images. Sometimes principal components are used to represent η as in (1.4), where the predicted parts are the least squares estimates of the observed y_i as predicted from the m largest principal components, and the residuals the estimates as predicted from the remaining $p-m$ principal components.

Without a further stipulation, definitions 1 and 2 are too broad to be of much use: given B which defines ξ we can use any full column rank matrix A to obtain a component decomposition as in definition 2, because ε, in each case, can simply be redefined as the difference $\eta - A\xi$. This means that the defining weights B alone do not suffice to specify a component decomposition uniquely. We therefore add

Definition 3. A component decomposition will be called a 'regression component decomposition' (RCD) iff the rows of the pattern A contain the regression weights for predicting the observed y_i from the components in ξ, i.e. iff A is of the form

$$A = \operatorname{cov}(\eta, \xi)\operatorname{var}^{-1}(\xi) \tag{1.5}$$

$$= \Sigma B \psi^{-1}. \tag{1.5a}$$

An A satisfying (1.5) and (1.5a) will be called a 'regression pattern'.

This definition retrieves some of the characteristics of the factor model, since the 'factor pattern' A^* in (1.1) is also the regression pattern for predicting the observed y_i from the implied common factors in ξ^*, as is well known.

A comparison of (1.1) with (1.4) shows that factor analysis and RCD share the same basic structural equation. Moreover, (1.5) also applies to both. The major differences between RCD and factor analysis result from our replacement of the implicit definition of ξ^*, ε^* of the factor model in (1.2) with the explicit definition of ξ, ε in (1.3).

2. Some simple consequences, and several equivalent characterizations

We list a number of straightforward consequences of the definitions (1.1)–(1.3) for future reference. Not all these consequences are new, and many are sufficiently straightforward that we can omit detailed proof. In some cases, we can also show sufficiency. Hence, a number of equivalent characterizations of the regression component decompositions defined by (1.3)–(1.5) can be obtained. In particular, the decomposition (1.3)–(1.5) will be shown to be equivalent to the decomposition implied by Guttman's (1952) 'Multiple Group Method'.

An immediate but important consequence of (1.3)–(1.5) is

$$\operatorname{cov}(\xi, \varepsilon) = \phi, \quad \text{for all } B \text{ in (1.3).} \tag{2.1}$$

This result, well known from standard regression algebra, follows immediately if we replace A in $\operatorname{cov}(\xi, \varepsilon) = \operatorname{cov}[B'\eta, (I-AB')\eta] = B'\Sigma - B'\Sigma BA'$ by its definition in (1.5).

In contrast to statistical treatments of component analysis, factor analysis begins with the extraction of the pattern A, since there is no B. It is therefore of interest to know how to express the matrix of defining weights B in terms of the regression pattern A. One finds

$$B = \Sigma^{-1} A(A'\Sigma^{-1} A)^{-1}, \quad \text{for all } A \text{ in (1.4), (1.5).} \tag{2.2}$$

Proof. From (1.5), $A'\Sigma^{-1} A = \psi^{-1} B'\Sigma(\Sigma^{-1})\Sigma B\psi^{-1}$, or

$$\psi = B'\Sigma B = (A'\Sigma^{-1} A)^{-1},$$

which can then be used to solve (1.5) for B in terms of A.

This means that one can use any full column rank matrix A, e.g. one obtained by any of the conventional factor extraction methods, to define a set of components for which this A is the regression pattern. If desired, one can replace the factor model (1.1)–(1.2) by a regression component decomposition which then has all the properties laid out here. It will later be shown that the so-called 'regression estimates' $\hat{\xi}^*$ of factor analysis do not have A^* in (1.1) as a regression pattern, since A^* is the regression pattern of ξ^*, not $\hat{\xi}^*$. Therefore (2.1) is violated for $\hat{\xi}^*$, although it is part of the definition of ξ^* in (1.2). Equation (2.2) implies that other factor 'estimates' can be computed which at least partially satisfy the stipulation of the factor model, if such are desired.

There are a number of other properties which regression component decompositions have in common with factor analysis. For example, (1.4), (2.1) imply at once

$$\operatorname{var}(\varepsilon) = \Sigma - A\psi A', \tag{2.3}$$

where A and B in (1.3) jointly satisfy

$$B'A = I. \tag{2.3a}$$

Proof. Using $\varepsilon = (I-AB')$ from (1.3) and (1.4) evaluate $\operatorname{var}(\varepsilon)$. Then apply (1.5). Equation (2.3a) follows from (2.2), and thus also from (1.3) to (1.5).

In conventional texts on factor analysis, the above simple result, rewritten as $\Sigma = A^*\psi A^{*\prime} + \operatorname{var}(\varepsilon^*)$, is usually called the 'fundamental theorem of factor analysis'. This prominence seems to us undeserved, as it follows as a trivial consequence of the uncorrelatedness condition (1.2). The full rank condition (1.2a) would seem to be more 'fundamental' since it implies the indeterminacy of the factors.

Upon straightforward evaluation, one further finds

$$\operatorname{cov}(\eta, \varepsilon) = \operatorname{var}(\varepsilon) \quad \text{for all } B, \tag{2.4}$$

and

$$\operatorname{cov}(\eta, A\xi) = \operatorname{var}(A\xi) \quad \text{for all } A \tag{2.5}$$

in exact analogy to the factor case, essentially as a direct consequence of the uncorrelatedness of ξ with ε in (2.1). As anticipated, the overlap in definitions of the factor model and RCD yields some similar properties at the variance–covariance level. In contrast, the rank property of the residual matrix, $\text{var}(\varepsilon)$ serves to differentiate between factor analysis and RCD. In factor analysis, $\text{var}(\varepsilon^*)$ has full rank p by definition (1.2). In regression components analysis, it has deficient rank $p-m$, as a consequence of the explicit definition of ξ^* in (1.3). This is most easily seen upon rewriting this matrix in (2.3) in terms of B, using (1.5):

$$\text{var}(\varepsilon) = \Sigma - \Sigma B(B'\Sigma B)^{-1} B'\Sigma. \tag{2.6}$$

This form of the residual variance–covariance matrix establishes the close connection of RCD with Guttman's (1952) 'Multiple Group Method'. Guttman showed that this matrix difference has rank $p-m$ if Σ had rank p and B full column rank m. He thus provided, to our knowledge for the first time, an explicit justification why the then popular extraction algorithms, such as the centroid method, method of triangular factoring, and principal axes method, 'work' (i.e. indeed bring about the desired rank reduction of the 'reduced' variance–covariance matrix $\Sigma - U^2$). Guttman further showed that a repetition of the same process on $\text{var}(\varepsilon)$, instead of Σ, under choice of a second weight matrix B^*, say, leads to a second set of components which will be uncorrelated with the first. In the limiting case, when each B has exactly one column, one thus obtains a set of mutually uncorrelated components. Guttman's perspective at the time was somewhat different from ours here: he discussed these developments in the context of factor analysis, where the decomposition is applied, implicitly, to the 'common parts' $\hat{\eta}^* = A^* \xi^*$ of the factor model. These common parts are not only unobserved, but also indeterminate, because ξ^* is. In contrast, our present focus is on component analysis, where the decomposition is applied to manifest, and thus uniquely defined, variables—precisely because we do not wish to become entangled with indeterminate random variables.

In passing, we note that it is the full rank condition (1.2a) of the factor model, not the diagonality condition $\text{var}(\varepsilon^*) = U^2$, diagonal, positive definite, which is at the heart of the factor indeterminacy problem. One is left with the same indeterminacy if the unique factors are correlated, as long as their variance–covariance matrix is nonsingular.

Finally, we note a connection between these developments and the theory of projectors. We can state in general that

$$\hat{\eta} = A\xi = P\eta, \quad \text{for some } P \text{ which satisfies} \tag{2.7}$$

$$P\Sigma B = \Sigma B, \quad \text{for all } B, \tag{2.7a}$$

and, alternatively,

$$\varepsilon = Q\eta, \text{ for some } Q \text{ which satisfies} \tag{2.8}$$

$$Q\Sigma B = \phi, \quad \text{for all } B. \tag{2.8a}$$

To prove these results, we note that equation (1.4a) can be rewritten

$$\eta = P\eta + (I-P)\eta = P\eta + Q\eta, \tag{2.9a}$$

where

$$P = AB' \tag{2.9b}$$

and

$$Q = I - AB'. \tag{2.9c}$$

Now (2.2) implies

$$A'B = I. \tag{2.10}$$

Hence,

$$P = P^2 = AB', \quad Q = (I-P) = Q^2. \tag{2.11}$$

That is, P and Q are (in general oblique) projectors. P can be written

$$P = \Sigma B(B'\Sigma B)^{-1}B' = A(A'\Sigma A)^{-1}A'\Sigma^{-1}, \tag{2.12}$$

so that

$$P\Sigma = \Sigma P', \tag{2.13}$$

and

$$\left.\begin{array}{ll} P\Sigma B = \Sigma B, & Q\Sigma B = \phi, \\ PA = A, & QA = \phi. \end{array}\right\} \tag{2.14}$$

P is an oblique projector for the column space of ΣB, or, equivalently, for the column space of A.

We now pause to state a more general conclusion.

Theorem 1. The triples [(1.3), (1.4), (1.5)] and [(1.3), (1.4), 2.k)] $k = 1, 2, \ldots, 8$ are nine equivalent characterizations of regression component decompositions, as defined.

Proof. See Appendix.

Theorem 1 shows that the formal structure of regression components analysis can be derived from a number of different sets of premises. These different premises provide different theoretical perspectives on the relation between factor analysis and RCD, while still leading to the same result. Some of the characterizations in Theorem 1 are useful in contrasting factor analysis with RCD. Those involving B, for example, as well as those involving the projectors P or Q, have no analogue in factor analysis simply because there exists no B which could be used to represent the factors ξ^*, ε^*.

Theorem 2. The factors of factor analysis are not components in the sense of definition 1.

Proof. If they were, their joint variance–covariance matrix could be written

$$\operatorname{var}\begin{pmatrix} \xi^* \\ \varepsilon^* \end{pmatrix} = \operatorname{var}\begin{Bmatrix} B'\eta \\ (I-AB')\eta \end{Bmatrix} = \begin{pmatrix} B' \\ I-AB' \end{pmatrix}\Sigma\begin{pmatrix} B & I-BA' \end{pmatrix} \tag{2.15}$$

for some B, which has rank $r \leqslant p$, since the rank of a product cannot exceed the rank of any of the factors. However, this conflicts with the full rank assumption (1.2a) of the factor model. Q.E.D.

A major difference, then, between RCD and factor analysis is that in RCD ξ, ε can be defined uniquely in terms of B, η whereas ξ^*, ε^* in the factor model cannot be defined uniquely in terms of η, A, ψ, U^2, since there is no B which could be used to define them uniquely as linear combinations of η. The representation in terms of projectors serves to distinguish between Guttman's 'images' and the regression components defined by (1.3)–(1.5). Although Guttman's (partial) 'images' and 'anti-images' are components in the sense of definition 1, an image is not a regression component because the weight matrices which define $A\xi$, ε are not idempotent.

Another distinction between regression components and the factors of factor analysis is reflected in the matrix of partial covariances

$$\psi - \psi A' \Sigma^{-1} A \psi = \operatorname{var}(\xi \mid \eta). \tag{2.16}$$

This matrix is zero in regression component analysis. This is intuitively reasonable, since we defined the components as linear combinations of the observed variables. Partialling out the observed variables from the components should then produce a residual variance–covariance matrix that is zero. In contrast, this matrix is not zero in factor analysis. There it is used in the formulas for constructing the indeterminate factors ξ^*, with $\operatorname{var}(\xi^*) = \psi^*$. These formulas, developed for the single factor case by Piaggio (1931), and later extended to the multiple factor case by Kestelman (1952) and Guttman (1955), state that for any factor pattern A^*, and a set of observed variables η, we can construct ξ^*, ε^* satisfying (1.1), (1.2) and (1.2a) as

$$\xi^* = \psi^* A^{*\prime} \Sigma^{-1} \eta + K\sigma, \quad \varepsilon^* = U^2 \Sigma^{-1} \eta - K\sigma, \tag{2.17}$$

where the matrix K is a Gram factor of $\operatorname{var}(\xi^* \mid \eta)$ in (2.16), i.e. satisfies

$$KK' = \operatorname{var}(\xi^* \mid \eta) = \psi^* - \psi^* A^{*\prime} \Sigma^{-1} A^* \psi^*. \tag{2.18}$$

The vector of random variables σ is subject only to the mild restrictions

$$\operatorname{var}(\sigma) = I_m, \quad \operatorname{cov}(\eta, \sigma) = \phi. \tag{2.19}$$

These restrictions leave a great deal of freedom for the choice of σ, which is the reason for the lack of determinacy of the implied variables ξ^*, ε^* of the factor model.

It is important to realize that this indeterminacy of ξ^* in the factor model differs from a simple lack of uniqueness which can be removed by suitable choice of one of the parameters of the model. It is clear, for example, that regardless of the factor indeterminacy, (1.1) implies that any nonsingular matrix T can be interpolated to redefine both A^*, ξ^* so that $A^{**} = A^* T$, $\xi^{**} = T^{-1} \xi^*$ will fit the model if and only if A^*, ξ^* do. This 'rotational indeterminacy' is generally not considered to be a serious problem, because it can be removed by simply selecting one of the parameters of the factor model, A^*, in some optimal way, e.g. by specifying that A^* be of 'simple structure'. The problem is that in the factor model this choice still leaves ξ^* indeterminate, i.e. infinitely many assignments for ξ^* exist which satisfy the model equally well for the same choice of A^*. In RCD, this rotational indeterminacy also exists, but there a choice of A, e.g. one of simple structure, fixes ξ since B is then uniquely determined by (2.2).

In passing, we note that this second uniqueness problem, the joint rotational indeterminacy of A, ξ, which exists in both cases, can be resolved in the RCD case in two alternative ways. Either one can rotate the pattern A to simple structure, or, alternatively, one can rotate the matrix of defining weights B to simple structure. This second alternative does not exist in the factor case. Yet, it seems to have at least as much intuitive appeal as the first: since one usually has some knowledge of the nature of the observed tests y_i in η, it is not unreasonable to seek components ξ which are parsimoniously defined in terms of these observed y_i, and which then can be interpreted in terms of the known y_i. In factor analysis, where the second alternative is not available, one usually reasons that the observed y_i should be explained as parsimonious functions of the unobserved and, hence, unknown x_j^* in ξ^*. This reasoning loses some of its stringency once it is realized that the x_j^* which are used to explain the manifest y_i are in fact not uniquely definable. Regrettably, this has usually not been realized in conventional treatments of this rotational indeterminacy, which thus is also seen to be affected directly by the long ignored factor indeterminacy issue.

The vanishing of $\operatorname{var}(\xi\,|\,\eta)$ in regression component analysis is both necessary and sufficient for obtaining a Gramian matrix $\Sigma - A\psi A'$ of rank $p-m$:

Theorem 3. Given two Gramian positive definite matrices of full rank p and m $(\leqslant p)$ respectively and a full column rank matrix A of order $p \times m$, the difference $\Sigma - A\psi A'$ is Gramian and of rank $p-m$ iff $\psi - \psi A'\Sigma^{-1}A\psi = \phi$, i.e.

$$\psi = (A'\Sigma^{-1}A)^{-1}.$$

Proof. If $\psi A'\Sigma^{-1}A\psi = \psi$, then

$$\Sigma - A\psi A' = \Sigma - 2A\psi A' + A(\psi A'\Sigma^{-1}A\psi)A' = (I - A\psi A'\Sigma^{-1})\Sigma(I - \Sigma^{-1}A\psi A')$$

is Gramian and of rank $p-m$, since $P = I - A\psi A'\Sigma^{-1} = P^2$ and has rank $p-m$. Conversely, if $\Sigma - A\psi A' = MM'$ for some $p \times (p-m)$ full column rank M, then

$$\Sigma = (A\psi^{\frac{1}{2}}, M)\begin{pmatrix} \psi^{\frac{1}{2}}A' \\ M' \end{pmatrix},$$

where $\psi^{\frac{1}{2}}$ is a symmetric Gram factor such that $\psi^{\frac{1}{2}}\psi^{\frac{1}{2}} = \psi$; $\psi^{\frac{1}{2}} = \psi^{\frac{1}{2}\prime}$. Let $VD^2V' = \Sigma$ be the eigendecomposition of Σ. Then there exists an orthogonal T such that $(A\psi^{\frac{1}{2}};\, M)\,T = VD$. Hence,

$$T'\begin{pmatrix} \psi^{\frac{1}{2}}A' \\ M' \end{pmatrix}\Sigma^{-1}(A\psi^{\frac{1}{2}};\, M)\,T = DV'(VD^{-2}V')\,VD = I,$$

that is,

$$\psi^{\frac{1}{2}}A'\Sigma^{-1}A\psi^{\frac{1}{2}} = I \quad \text{or} \quad A'\Sigma^{-1}A = \psi^{-1}.$$

Q.E.D.

Before turning to a comparison of such RCDs with the conventional practice of factor analysis, we state a simple lemma which will be useful later.

Lemma. A is the regression pattern for predicting η from ξ iff $A_s = SAT$ is the regression pattern for predicting $\eta^* = S\eta$ from $\xi^* = T^{-1}\xi$ for all non-singular S, T.

Proof. The result is a direct consequence of (1.5) applied to η^*, ξ noting that $\operatorname{cov}(\eta^*, \xi^*) = S \operatorname{cov}(\eta, \xi)\, T^{-1}$.

3. Falsifiability and some implications for 'factor score estimation'

A comparison between factor analysis and RCDs, as developed so far, might seem to favor the latter: at the covariance level there are a number of similarities, and at the random variable level the main difference is that regression components are uniquely defined linear combinations of the manifest variables, whereas the factors of the factor model are not. Aside from all theory, it should also be clear that RCDs are more efficient computationally, because, in general, they do not require estimation of the communalities of the factor model. In factor analysis, lengthy iterative procedures are necessary to obtain optimal estimates for these variance parameters of the model. One of the better understood and currently popular estimation methods is the maximum likelihood procedure developed by Lawley (1940) and perfected by Howe (1955), Jöreskog (1967) and others. Maximum likelihood methods require a prior guess for m, the number of common factors. Upon convergence, this hypothesis can be tested statistically. If it has to be rejected for the data on hand, the iterative process has to be repeated for a revised m. Schönemann & Wang (1972) found that the number of common factors m so arrived at is usually larger than had been assumed or found practical in previous analyses of the same data by earlier, technically inferior extraction methods. In other words, if treated statistically, the factor model does not fit as often and as well as had been claimed in the days when no rigorous methods for testing the fit were available. Schönemann & Wang also found that simply raising m does not resolve all problems in practical applications of the factor model because it often invites new troubles. As the number of factors is raised in an effort to improve the statistical fit, one usually finds that the identifiability for some of the variance parameters in U^2 is lost. This hazard, though well known in theory (e.g. Wilson, 1939; Anderson & Rubin, 1956), has not been of any greater concern to most practitioners of factor analysis than the equally distressing factor indeterminacy issue, and most 'classical' texts on factor analysis ignore both.

When compared to component analysis, factor analysis appears to have many disadvantages. Yet, proponents of the factor model have repeatedly characterized component analysis as a mere 'approximation to the factor model'. This view is based in large part on the fact that most versions of component analysis which have been investigated are tautological, whereas the factor model is, at least in principle, falsifiable. This aspect of factor analysis was quite explicit in Spearman's days, but it became more and more remote in the Thurstone era.

It has now re-emerged through the development of statistical algorithms for testing some of the hypotheses implied by the factor model. We now have some well-understood and computationally manageable algorithms for fitting the factor model statistically. This, it has been said, is an undeniable advantage over any form of component analysis, which is simply a tautological description of the data.

This fairly widespread belief is only half true, as we shall now show: it is true that the factor model, by definition, is falsifiable at the covariance level, for any fixed m ($<p-1$). It is also true that many forms of component analysis, as usually treated, are tautological. But it is not true that this is a critical difference between both methods. The factor model becomes falsifiable by specifying a diagonal variance–covariance matrix for the unique factors, i.e. by specifying one of the parameters of the model partially. There is no obvious reason why a similar specification could not also be imposed on component decompositions, thereby rendering them falsifiable. The question of whether the unobserved ('latent') random variables are indeterminate or not is unrelated to this falsifiability issue. As we shall show, if one does not wish to entertain a model with indeterminate random variables, for the description of data, then one can always replace it by an empirically equivalent regression component model with unambiguously defined random variables. To establish this result we need:

Theorem 4. $\Sigma = A^* A^{*\prime} + U^2$, with A^* of full column rank m and U^2 positive definite, diagonal, iff there exists a diagonal, positive definite matrix U such that $\Sigma^* = U^{-1}\Sigma U^{-1} = AA' + E$, where AA' has latent roots $b_1^2 \geqslant b_2^2 \geqslant \ldots \geqslant b_m^2 > 1$, and $E = E' = E^2$, $EA = \phi$.

Proof. Since A^* has full column rank m, $\Sigma = A^* A^{*\prime} + U^2$ implies that $\Sigma^* = U^{-1}\Sigma U^{-1} = (U^{-1}A^*)(A^{*\prime}U^{-1}) + I$ has $p-m$ roots equal to unity, and m roots $b_1^2, \ldots, b_m^2 > 1$. If the eigendecomposition of Σ^* is $\Sigma^* = L_1 D_{b1}^2 L_1' + L_2 L_2'$, where the m largest roots are in D_{b1}^2, and (L_1, L_2) contains the orthogonalized eigenvectors, then we can set $A = L_1 D_{b1}$, $E = L_2 L_2'$, to obtain

$$\Sigma^* = U^{-1}\Sigma U^{-1} = AA' + E,$$

with $E = E' = E^2$.

Conversely, if a positive definite diagonal matrix U exists such that

$$\Sigma^* = U^{-1}\Sigma U^{-1} = AA' + E,$$

and E satisfies $E = E' = E^2$, then we can write $E = L_2 L_2'$, for some L_2 which satisfies $L_2' L_2 = I_{p-m}$. $EA = L_2 L_2' A = \phi$ then implies that we can write $A = L_1 T$ for some nonsingular $m \times m$ matrix T, where L_1 is the orthogonal complement of L_2, and an orthogonal basis for the column space of A, so that $L_1' L_2 = \phi$, $L_1' L_1 = I_m$. Thus, we have

$$\Sigma^* = AA' + E = L_1 TT' L_1' + L_2 L_2' = L_1(TT' - I) L_1' + L_1 L_1' + L_2 L_2' = BB' + I.$$

$B = L_1(TT'-I)^{\frac{1}{2}}$ exists and is of full column rank m since all nonzero roots of $AA' = L_1 TT' L_1'$ are larger than unity. Hence, $\Sigma = U\Sigma^* U = A^* A^{*\prime} + U^2$, where $A^* = UB$ of full column rank m, and U^2 is diagonal, positive definite. Q.E.D.

Theorem 4 implies:

Theorem 5. The factor model (1.1)–(1.2) fits a given η iff a regression component decomposition (1.3)–(1.5) fits the rescaled variables

$$\eta^* = U^{-1}\eta \tag{3.1}$$

for some positive definite diagonal matrix U so that the $p-m$ nonzero eigenvalues of the residual variance–covariance matrix

$$E = \mathrm{var}(\varepsilon), \quad \text{for } \varepsilon = \eta^* - A\xi \tag{3.2}$$

are all equal to unity, or, equivalently, iff a RCD $\eta = A_s\xi + \varepsilon_s$ fits $\eta = U\eta^*$ where $\varepsilon_s = U\varepsilon$ for ε in (3.2).

Proof. If the factor model fits, $\Sigma = A^*A^{*\prime} + U^2$. Hence, by Theorem 4, $\Sigma^* = U^{-1}\Sigma U^{-1} = AA' + E$, with $E = E' = E^2$. By Theorem 1, (1.3), (1.4), (2.3) suffice to characterize a regression component decomposition for some B that satisfies $B'A = I$. Interpreting A in $\Sigma^* = AA' + E$ as the regression pattern for $\xi = B'\eta^*$ with $\mathrm{var}(\xi) = \psi = I_m$ and E as $\mathrm{var}(\varepsilon)$ for $\varepsilon = (I - AB')\eta^*$ uniquely defines a regression decomposition on the rescaled variables $\eta^* = U^{-1}\eta$, for $B = \Sigma^{*-1}A(A\Sigma^{*-1}A)^{-1}$ by (2.2). The $p-m$ nonzero roots of $\mathrm{var}(\varepsilon)$ are unity, since $E = E' = E^2$.

Conversely, if a diagonal positive definite matrix U exists such that the residual variance–covariance matrix satisfies $E = E' = E^2$, then, by Theorem 4, $\Sigma = U\Sigma^*U$ must be of the form $\Sigma = A^*A^{*\prime} + U^2$. Hence the factor model (1.1), (1.2) fits for same ξ^*, ε^* with $\mathrm{var}(\xi^*) = \psi = I_m$. Finally, if (3.2) is a RCD for (3.1), then $\eta = (UA)\xi + (U\varepsilon) = A_s\xi + \varepsilon_s$ is a RCD for η, and conversely, by Lemma 1. Q.E.D.

The point is simply that the factor model can only be falsified at the variance–covariance level, not at the random variable level. The implied structure of the random variables of the factor model is empirically empty. It can be replaced, if desired, by an equally empirically empty component structure which has none of the logical problems of factor scores and their 'estimates'.

The component model in Theorem 5 differs from a tautological regression decomposition in terms of the added stipulations (3.1) and (3.2). The falsifiable hypothesis is that a suitably chosen rescaling of the observed variables y_i can be found which yields an idempotent variance–covariance matrix for the residuals in ε. This hypothesis need not be true—indeed, for small m, it is probably false in most cases. The factor model holds if, and only if, this hypothesis is true.

Some statistical treatments of principal components analysis have recommended that component extraction be halted when the eigenvalues become indistinguishable (e.g. Kendall, 1961). The regression component decomposition of Theorem 5 follows the spirit of these recommendations, because the indistinguishable components are, in effect, discarded as (correlated) error contained in ε.

It is the diagonality condition of the factor model, $\mathrm{var}(\varepsilon^*) = U^2 =$ diagonal which yields its falsifiability. The same condition also yields falsifiability for the component decomposition (1.3)–(1.5), (3.1), (3.2). The full rank condition (1.2a), on the other hand, yields the indeterminacy of the random variables of the factor model. This indeterminacy remains even if the diagonality condition is weakened—so long as the full rank condition is retained. In short, falsifiability and factor indeterminacy are unrelated. We can retain one and eliminate the other.

Hence, any time the factor model with m common factors fits η, there is also a regression component decomposition with m components which fits η. In using this RCD, one defines the latent variables in ξ uniquely as known, readily interpretable linear combinations of η. These latent variables, unlike their common factor counterparts, are determinate. Moreover, as shown earlier, they satisfy a number of the variance–covariance properties of the factor model, i.e. (2.1), (2.3), (2.4), (2.5). The RCD accomplishes the essential purpose of factor analysis with less computational effort, and a considerable gain in conceptual clarity. To appreciate this more fully, let us compare the properties of such an RCD with the analogous procedure in factor analysis, i.e. the 'estimation' of factor scores via the regression method. These 'regression estimates' are simply components in the sense of definition 1 computed as

$$\hat{\xi}^* = \psi^* A^{*\prime} \Sigma^{-1} \eta. \tag{3.3}$$

For convenience, and without loss of generality, we will assume that

$$\operatorname{var}(\xi^*) = \psi^* = I_m. \tag{3.3a}$$

We will temporarily ignore the semantic problems inherent in the use of the term 'estimates', and simply examine whether they satisfy the various constraints of the factor model (1.1)–(1.2a). The regression pattern for the regression 'estimates' is

$$A^{**} = \operatorname{cov}(\eta, \hat{\xi}^*) \operatorname{var}^{-1}(\hat{\xi}^*) = A^*(A^{*\prime}\Sigma^{-1}A^*)^{-1} \neq A^*. \tag{3.4}$$

Defining the residual $\hat{\varepsilon}^*$ tautologically as

$$\hat{\varepsilon}^* = \eta - A^* \hat{\xi}^* \tag{3.5}$$

one finds

$$\operatorname{var}(\hat{\varepsilon}^*) = U^2 \Sigma^{-1} U^2 \neq U^2 = \text{diagonal} = \operatorname{var}(\varepsilon^*), \tag{3.6}$$

and also

$$\operatorname{cov}(\hat{\xi}^*, \hat{\varepsilon}^*) = A^{*\prime}\Sigma^{-1} U^2 \neq \phi = \operatorname{cov}(\xi^*, \varepsilon^*), \tag{3.7}$$

$$\operatorname{var}(\hat{\xi}^*) = A'^{*}\Sigma^{-1} A^* \neq I_m = \operatorname{var}(\xi^*). \tag{3.8}$$

In short, the 'regression estimates' have none of the properties of the 'factors' which they are supposedly estimating. The regression components, on the other hand, satisfy several of the covariance properties of the factors, namely (2.1), (2.3), (2.4), (2.5), and they satisfy all the properties of the empirically indistinguishable regression component model.

4. Conclusions

To avoid misunderstanding, we emphasize that we do not recommend the use of the falsifiable component model in Theorem 5 in practical work. We believe that factor analysis has achieved its erstwhile popularity not because of the properties of the factor model, but, rather, in spite of them. The status of factor analysis as a falsifiable scientific hypothesis has rarely been taken seriously since Thurstone popularized multiple factor analysis as a general research tool. It has not been taken seriously as a model by those theoreticians who urged its widespread use, and went to great lengths to ignore the theoretical problems attending to this model. It has not been taken seriously by most users, who were generally uninformed about the defects of the factor model. They were left with the erroneous impression that the factor model 'always worked', for a conveniently small number of common factors. They were told to 'estimate factor scores' because 'factor scores cannot be computed, they can only be estimated'. This myth was perpetuated through decades, although it is obviously false, as could have been known since Wilson (1928).

In using 'factor score estimates' the user was in effect employing components, albeit components which were derived in an illogical and cumbersome way, and which, as we have shown, share none of the properties of the random variables defined by the model.

As we have demonstrated, there is no need for factor theory at all. Whether factor analysis is used solely as a data reduction technique, or whether it is taken seriously as a model for the variance–covariance matrix of the observed variables, it can always be replaced by regression component analysis with no loss in flexibility and potential for falsifiability, and with considerable gain in conceptual clarity and computational efficiency.

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Appendix

Proof of Theorem 1

Since necessity has already been shown where it was not immediately obvious, we need to show sufficiency to establish the equivalence. For the sake of brevity, we shall write $(2.k)$: $k = 1, 2, \ldots, 8$ to mean $[(1.3), (1.4)\ (2.k)] \Rightarrow [(1.3).(1.4), (1.5)]$.

(2.1): $(1.3) \Rightarrow \operatorname{cov}(\eta, \xi) = \Sigma B$. $(1.4), (2.1) \Rightarrow \operatorname{cov}(\eta, \xi) = A\psi$. Hence $A = \Sigma B \psi^{-1}$.

(2.2): (1.3), $(2.2) \Rightarrow \psi = B'\Sigma B = (A'\Sigma^{-1}A)^{-1}$. Substitution in (2.2) gives $A = \Sigma B \psi^{-1}$.

(2.3): $(1.3), (1.4), (2.3) \Rightarrow B'\varepsilon = \phi$. Hence, from (2.3), $\operatorname{var}(\varepsilon) B = \Sigma B - A\psi = \phi$ or $A = \Sigma B \psi^{-1}$.

(2.4): (1.4), $(2.4) \Rightarrow \operatorname{cov}(\eta, \eta - A\xi) = \operatorname{var}(\eta - A\xi) \Leftrightarrow \Sigma - \operatorname{cov}(\eta, \xi) A' = \Sigma - \operatorname{cov}(\eta, \xi) A' - A \operatorname{cov}(\xi, \eta) + A\psi A'$, or $A \operatorname{cov}(\xi, \eta) = A\psi A'$. $(1.3 \Rightarrow AB'\Sigma = A\psi A'$. Rank $(A) = m \Rightarrow A = \Sigma B \psi^{-1}$.

(2.5): $(1.3), (2.5) \Rightarrow \Sigma BA' = A\psi A$. Rank $(A) = m \Rightarrow A = \Sigma B \psi^{-1}$.

(2.6): $(1.3), (1.4) \Rightarrow \operatorname{var}(\varepsilon) = \Sigma - AB'\Sigma - \Sigma BA' + AB'\Sigma BA'$. $(2.6) \Rightarrow$
$\Sigma B(B'\Sigma B)^{-1} B'\Sigma - AB'\Sigma - \Sigma BA' + A(B'\Sigma B) A' = \phi \Leftrightarrow$
$[\Sigma B(B'\Sigma B)^{-1} - A](B'\Sigma B)[\Sigma B(B'\Sigma B)^{-1} - A] = \phi$. $\psi = B'\Sigma B$, n.s.
$\Rightarrow A = \Sigma B \psi^{-1}$.

(2.7): $(2.7a) \Rightarrow A\psi = P \operatorname{cov}(\eta, \xi)$. $(1.3) \Rightarrow A\psi = P\Sigma B$. $(2.7a) \Rightarrow A\psi = \Sigma B$, hence $A = \Sigma B \psi^{-1}$. Note that the idempotency of P is not needed.

(2.8): $(1.4), (2.8) \Rightarrow \eta - A\xi = Q\eta \Rightarrow \Sigma - A \operatorname{cov}(\xi, \eta) = Q\Sigma$. $(1.3) \Rightarrow \Sigma - AB'\Sigma = Q\Sigma \Rightarrow \Sigma B - AB'\Sigma B = Q\Sigma B$. $(2.8a) \Rightarrow \Sigma B = A\psi$. Hence $A = \Sigma B \psi^{-1}$. Again, the idempotency of Q is not needed.